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# Hankel Operators and Gramians for Nonlinear Systems

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## Abstract

In the theory for continuous-time linear systems, the system Hankel operator plays an important role in a number of realization problems ranging from providing an abstract notion of *state* to yielding tests for state space minimality and algorithms for model reduction. But in the case of continuous-time nonlinear systems, Hankel theory is considerably less developed beyond a well known Hankel *mapping* introduced by Fliess in 1974. In this paper, a definition of a system Hankel operator is developed for causal  $L_2$ -stable input-output systems. If a generating series representation of the input-output system is given then an explicit representation of the corresponding Hankel operator is possible. If, in addition, an affine state space model is available with certain stability properties then a unique factorization of the Hankel operator can be constructed with direct connections to well known and new nonlinear Gramian extensions.

## 1. Introduction

In the theory of continuous-time linear systems, the system Hankel operator plays an important role in a number of realization problems. For example, when viewed as a mapping from past inputs to future outputs, it plays a direct role in the abstract definition of *state* [9]. It also plays a central role in minimality theory and in model reduction problems. Specifically, the Hankel operator supplies a set of similarity invariants, the so called Hankel singular values, which can be used to quantify the importance of each state in the corresponding input-output system [8]. The Hankel operator can also be factored into the composition of an observability and controllability operator, from which Gramian matrices can be defined and the notion of a balanced realization follows [6, 8, 11].

In the case of continuous-time nonlinear systems, Hankel theory is much less developed, but important results do exist. The first result along these lines is due to Fliess [2]-[4] who used a system Hankel *mapping* to describe when an affine realization of an input-output system described by a generating series is minimal. This mapping in essence plays the same role that the system

Hankel *matrix* does in linear system theory. In quite a different setting, the notion of Hankel singular values was generalized to nonlinear systems by Scherpen in [13, 14, 15] and used in model reduction problems. Connections between minimality and these invariants were then introduced in [16, 17]. But to date, no exact analogue of the Hankel operator for a nonlinear system has been fully developed in the literature. In this paper we take first steps in this direction by first giving a definition of this concept for causal  $L_2$ -stable input-output systems and then supplying an explicit representation of it via generating series. Its connection to the Hankel mapping of Fliess is also established. When an affine state space realization is available with certain stability properties, a factorization of this operator is developed and related to the controllability and observability functions of Scherpen in [13, 14]. Thus, a link is established between the newly defined system Hankel operator and one nonlinear generalization of the notion of Gramian matrices. Finally a new generalization of the Gramian is introduced which in some sense is the most direct analogue of that which is used for linear systems.

The paper is organized as follows. In Section 2, we review the well known linear system definitions of the system Hankel matrix; Hankel operator; and controllability/observability operators, Gramians and functions. This is partly to establish notation, but also to motivate the particular approach taken in the nonlinear case. Then in Section 3, we briefly review well known results regarding the Hankel mapping of Fliess for any input-output system that can be represented by a generating series. Finally, in Section 4, we introduce a Hankel operator definition motivated by the linear system development given in Section 2 and the Hankel mapping of Section 3. We then develop its relationship to nonlinear Gramian extensions.

The mathematical notation used throughout is fairly standard. The inner product on  $\mathbb{R}^n$  is represented as  $\langle x, y \rangle = x^T y$ .  $L_2^i(a, b)$  represents the set of Lebesgue measurable functions,  $i$ -component vector-valued, with finite  $L_2$  norm,  $\| \cdot \|_{L_2}$ . The inner product on  $L_2^i(a, b)$  is denoted by

$$\langle f, g \rangle_{L_2} = \int_a^b f(t)^T g(t) dt.$$

We abbreviate  $L_2^i(-\infty, \infty)$  as  $L_2^i$ . If  $h$  is a differentiable function, and  $g$  is a vector field then  $L_g h$  denotes the Lie derivative of  $h$  with respect to  $g$ . The symbol  $\otimes$  denotes the matrix Kronecker product.

## 2. Linear Systems as a Paradigm

In this section we review the well known linear system definitions of the system Hankel matrix; Hankel operator; and controllability/observability operators, Gramians and functions. [6, 8, 9, 13, 14]. This is partly to establish notation, but also to motivate the particular approach taken in the nonlinear case. The relationships between these concepts are stated without proof since they are either standard or easily verified.

Consider a continuous-time, causal linear input-output system  $S : u \rightarrow y$  with impulse response  $H(t)$ . Let

$$H(t) = \sum_{k=0}^{\infty} H_{k+1} \frac{t^k}{k!}, \quad t \geq 0 \quad (1)$$

denote its Taylor series expansion about  $t = 0$  where  $H_k \in \mathbb{R}^{p \times m}$  for each  $k$ . The system Hankel matrix is defined as  $\mathcal{H} = [\mathcal{H}_{i,j}]$  where  $\mathcal{H}_{i,j} = H_{i+j-1}$  for  $i, j \geq 1$ . If  $S$  is also BIBO stable then the system Hankel operator is the well defined mapping

$$\begin{aligned} \hat{\mathcal{H}} &: L_2^m[0, +\infty) \rightarrow L_2^p[0, +\infty) \\ &: \hat{u} \rightarrow \hat{y}(t) = \int_0^\infty H(t+\tau) \hat{u}(\tau) d\tau. \end{aligned}$$

If we define the *time flipping* operator as

$$\begin{aligned} \mathcal{F} &: L_2^m[0, +\infty) \rightarrow L_2^m \\ &: \hat{u} \rightarrow u(t) = \begin{cases} \hat{u}(-t) & : t < 0 \\ 0 & : t \geq 0. \end{cases} \end{aligned}$$

then clearly  $\hat{\mathcal{H}}(\hat{u}) = (S \circ \mathcal{F})(\hat{u})$ .

The system Hankel matrix and operator can be related by the following formalism. Substituting the series (1) into the convolution integral representing  $S$  yields the equivalent iterated integral representation

$$y(t) = \sum_{k=0}^{\infty} H_{k+1} \underbrace{E_{0 \dots 0} 1}_k(t, -\infty)[u],$$

where

$$\underbrace{E_{0 \dots 0} 1}_k(t, t^0)[u] := \int_{t^0}^t \int_{t^0}^{\tau_1} \dots \int_{t^0}^{\tau_{k+1}} u(\tau_1) d\tau_1 d\tau_2 \dots d\tau_{k+1}.$$

For any  $\hat{u} \in L_2^m[0, \infty)$ , it is a straightforward calculation to compute the derivatives

$$\hat{y}^{(i)}(t) = \sum_{k=0}^{\infty} H_{i+k+1} \underbrace{E_{0 \dots 0} 1}_k(t, -\infty)[u], \quad i \geq 0, \quad t > 0,$$

when  $u(t) = \mathcal{F}(\hat{u})$ . In matrix notation, it then follows that

$$\underbrace{\begin{bmatrix} \hat{y}(t) \\ \hat{y}'(t) \\ \hat{y}''(t) \\ \vdots \end{bmatrix}}_{\hat{\mathcal{Y}}(t)} = \underbrace{\begin{bmatrix} H_1 & H_2 & H_3 & \dots \\ H_2 & H_3 & H_4 & \dots \\ H_3 & H_4 & H_5 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\mathcal{H}} \underbrace{\begin{bmatrix} E_1(t, -\infty)[u] \\ E_{01}(t, -\infty)[u] \\ E_{001}(t, -\infty)[u] \\ \vdots \end{bmatrix}}_{\mathcal{E}_1(t, -\infty)[u]}. \quad (2)$$

Given some  $t^* > 0$ , there exists an open interval  $I$  containing  $t^*$  such that for all  $t \in I$

$$\begin{aligned} \hat{y}(t) &= \sum_{i=0}^{\infty} \hat{y}^{(i)}(t^*) \frac{(t-t^*)^i}{i!} \\ &= \sum_{i=0}^{\infty} \hat{y}^{(i)}(t^*) \underbrace{E_{0 \dots 0} 1}_i(t, t^*) \\ &= \mathcal{E}_0^T(t, t^*) \hat{\mathcal{Y}}(t^*), \end{aligned}$$

where  $\underbrace{E_{0 \dots 0} 1}_i(t, t^*) := \frac{(t-t^*)^i}{i!}$  and

$$\mathcal{E}_0(t, t^*) := \begin{bmatrix} 1 \\ E_0(t, t^*) \\ E_{00}(t, t^*) \\ \vdots \end{bmatrix} \otimes I_p.$$

Thus, near  $t^*$

$$\hat{y}(t) = \hat{\mathcal{H}}(\hat{u}) = \mathcal{E}_0^T(t, t^*) \mathcal{H} \mathcal{E}_1(t^*, -\infty)[u]. \quad (3)$$

Let  $(A, B, C)$  be a state space realization of  $S$  with dimension  $n$ . Any such realization induces a factorization of the system Hankel matrix into the form  $\mathcal{H} = \mathcal{O}\mathcal{C}$ , where  $\mathcal{O}$  and  $\mathcal{C}$  are the (extended) observability and controllability matrices. If the realization is asymptotically stable then the Hankel operator can be written as the composition of uniquely determined observability and controllability operators; that is,  $\hat{\mathcal{H}} = \hat{\mathcal{O}}\hat{\mathcal{C}}$ , where the controllability and observability operators are defined as

$$\begin{aligned} \hat{\mathcal{C}} &: L_2^m[0, +\infty) \rightarrow \mathbb{R}^n : \hat{u} \rightarrow \int_0^\infty e^{At} B \hat{u}(t) dt \\ \hat{\mathcal{O}} &: \mathbb{R}^n \rightarrow L_2^p[0, +\infty) : x \rightarrow \hat{y}(t) = C e^{At} x. \end{aligned}$$

Since  $\hat{\mathcal{C}}$  and  $\hat{\mathcal{O}}$  have a finite dimensional range and domain, respectively, they are compact operators; and the composition  $\hat{\mathcal{O}}\hat{\mathcal{C}}$  is also a compact operator [10]. In light of equation (3), we also have the identities

$$\begin{aligned} \hat{\mathcal{C}}(\hat{u}) &= \lim_{t^* \rightarrow 0} \mathcal{C} \mathcal{E}_1(t^*, -\infty)[u] \\ \hat{\mathcal{O}}(x) &= \lim_{t^* \rightarrow 0} \mathcal{E}_0^T(t, t^*) \mathcal{O} x. \end{aligned}$$

From the definition of the (Hilbert) adjoint operator, it is easily shown that  $\hat{\mathcal{C}}$  and  $\hat{\mathcal{O}}$  have corresponding adjoints

$$\begin{aligned} \hat{\mathcal{C}}^* &: \mathbb{R}^n \rightarrow L_2^m[0, +\infty) : x \rightarrow B^T e^{A^T t} x \\ \hat{\mathcal{O}}^* &: L_2^p[0, +\infty) \rightarrow \mathbb{R}^n : y \rightarrow \int_0^\infty e^{A^T t} C^T y(t) dt. \end{aligned}$$

For any  $x_1, x_2 \in \mathbb{R}^n$ :

$$\begin{aligned} \langle x_1, \hat{C}^* x_2 \rangle &= x_1^T \int_0^\infty e^{At} B B^T e^{A^T t} dt x_2 \\ &:= x_1^T P x_2 \end{aligned} \quad (4)$$

$$\begin{aligned} \langle x_1, \hat{O}^* \hat{O} x_2 \rangle &= x_1^T \int_0^\infty e^{A^T t} C^T C e^{At} dt x_2 \\ &:= x_1^T Q x_2. \end{aligned} \quad (5)$$

Of course,  $P$  and  $Q$  are the usual controllability and observability Gramian matrices. Now consider the following definition for the *energy functions* first described by Scherpen for general affine nonlinear systems [13, 14].

**Definition 2.1** *The controllability and observability functions for the system  $(A, B, C)$  are defined, respectively, as*

$$L_c(x) = \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty)=0, x(0)=x}} \frac{1}{2} \int_{-\infty}^0 \|u(t)\|^2 dt$$

and

$$L_o(x) = \frac{1}{2} \int_0^\infty \|y(t)\|^2 dt,$$

when  $x(0) = x$ , and  $u(t) = 0$  for  $0 \leq t < \infty$ .

Clearly,  $L_c(x)$  has the interpretation of being the minimum amount of input energy required to drive the system from zero at  $t = -\infty$  to  $x(0) = x$ , while  $L_o(x)$  is equivalent to the energy generated by the natural response of the system to  $x(0) = x$ . These functions need not be well defined for all  $x \in \mathbb{R}^n$ . For example,  $L_o$  may not be finite if the system is not asymptotically stable, or  $L_c$  may not be finite if the state  $x$  is not reachable. If the system is reachable and asymptotically stable, then in light of equations (4) and (5), it can be shown directly that

$$L_c(x) = \frac{1}{2} x^T P^{-1} x = \frac{1}{2} \langle x, (\hat{C}^*)^{-1} x \rangle \quad (6)$$

$$L_o(x) = \frac{1}{2} x^T Q x = \frac{1}{2} \langle x, (\hat{O}^* \hat{O}) x \rangle. \quad (7)$$

### 3. Hankel Mappings for Nonlinear Systems

In this section we briefly review well known results regarding Hankel mappings for any input-output system that can be represented by a generating series (Chen-Fliess functional expansion). The definition is due to Fliess [2]-[4], and a detailed treatment of this material may also be found in [7]. (See also [5].)

Let  $S$  be a given input-output map represented by a convergent generating series

$$S : u \rightarrow y(t) = \sum_{\eta \in I^*} c(\eta) E_\eta(t, t_0)[u], \quad t \geq t_0,$$

where  $I^*$  is the set of multi-indices for the index set  $I = \{0, 1, \dots, m\}$ ,  $c(\eta) \in \mathbb{R}^p$ , and

$$E_{i_k \dots i_0}(t, t_0)[u] = \int_{t_0}^t u_{i_k}(\tau) E_{i_{k-1} \dots i_0}(\tau, t_0)[u] d\tau$$

with  $E_\emptyset(t, t_0)[u] := 1$  and  $u_0(t) := 1$ . The mapping  $S$  can then also be represented by a formal power series in noncommuting monomials  $\mathcal{Z} = \{z_0, z_1, \dots, z_m\}$  via

$$c = \sum_{\eta \in I^*} c(\eta) z_\eta,$$

where  $z_\eta := z_{i_k} \dots z_{i_0}$  when  $\eta = (i_k \dots i_0)$ . Now define the sets:

$\mathbb{R} \langle \mathcal{Z} \rangle$  : the set of polynomials in  $\mathcal{Z}$  over  $\mathbb{R}$ ;

$\mathbb{R}^p \ll \mathcal{Z} \gg$  : the set of formal power series in  $\mathcal{Z}$  over  $\mathbb{R}^p$ .

**Definition 3.1** *The Hankel mapping associated with a formal power series  $c$  is the  $\mathbb{R}$ -vector space morphism*

$$\mathcal{H} : \mathbb{R} \langle \mathcal{Z} \rangle \rightarrow \mathbb{R}^p \ll \mathcal{Z} \gg,$$

*uniquely specified by the generalized shifting property*

$$[\mathcal{H}(z_\zeta)](\gamma) = c(\gamma \zeta),$$

for any  $\gamma, \zeta \in I^*$ .

Let  $\{\zeta_i\}_{i>0}$  denote a re-indexing of the elements of  $I^*$  via the natural numbers. Let  $\{\gamma_j\}_{j>0}$  denote another such re-indexing of  $I^*$ , possibly distinct from the first. In matrix notation then, the equation  $s = \mathcal{H}(p)$  can be represented as

$$\begin{bmatrix} s_{\gamma_1} \\ s_{\gamma_2} \\ s_{\gamma_3} \\ \vdots \end{bmatrix} = \begin{bmatrix} c(\gamma_1 \zeta_1) & c(\gamma_1 \zeta_2) & c(\gamma_1 \zeta_3) & \dots \\ c(\gamma_2 \zeta_1) & c(\gamma_2 \zeta_2) & c(\gamma_2 \zeta_3) & \dots \\ c(\gamma_3 \zeta_1) & c(\gamma_3 \zeta_2) & c(\gamma_3 \zeta_3) & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} p_{\zeta_1} \\ p_{\zeta_2} \\ p_{\zeta_3} \\ \vdots \end{bmatrix},$$

where  $p = \sum_i p_{\zeta_i} z_{\zeta_i}$  and  $s = \sum_j s_{\gamma_j} z_{\gamma_j}$ . In this context we have the following definition.

**Definition 3.2** *The Lie rank of a formal power series  $c$  is defined as  $\rho_L(c) := \dim(\mathcal{H}(\mathcal{L}(\mathcal{Z})))$ , where  $\mathcal{L}(\mathcal{Z})$  denotes the smallest Lie algebra containing  $\mathcal{Z}$ .*

Let  $M$  be an  $n$ -dimensional analytic state space manifold, and let

$$\begin{aligned} \dot{x} &= f(x) + g(x) \\ y &= h(x) \end{aligned}$$

be a system defined in terms of local coordinates on  $M$ . We assume that  $f$ ,  $g$ , and  $h$  are analytic on  $M$ . A realization  $(f, g, h)$  defined locally about  $x^o \in M$  is said to *realize* a formal power series  $c$  if for every  $\eta = (i_k \dots i_0) \in I^*$

$$\begin{aligned} c(\eta) &= L_{g_\eta} h(x^o) \\ &:= L_{g_{i_0}} L_{g_{i_1}} \dots L_{g_{i_k}} h(x^o), \end{aligned} \quad (8)$$

where  $g_0 := f$  and  $g_i$  is the  $i$ th column of  $g$  when  $i > 0$ . It is well known that if a certain growth condition on the coefficients  $\{c(\eta)\}_{\eta \in I^*}$  is satisfied, then there exists a realization of  $c$  if and only if the Lie rank of  $c$  is finite. The following result characterizes minimality.

**Theorem 3.1** *An analytic realization  $(f, g, h)$  about  $x^o \in M$  of a formal power series  $c$  is minimal if and only if its dimension is equal to the Lie rank  $\rho_L(c)$ .*

#### 4. Hankel Operators for Nonlinear Systems

In this section we first introduce a system Hankel operator motivated by the linear case described in Section 2. We then relate it to the Hankel mapping of Fliess from the previous section. Finally, we show that as in the linear case this Hankel operator has a natural factorization into a controllability and observability operator pair which can be related to the energy functions through an appropriate generalization of the adjoint operator.

In the following development we use the convention that  $L_2$ -stability of an input-output system,  $S$ , means that  $u \in L_2^m(-\infty, 0]$  implies that  $S(u)$  restricted to  $[0, +\infty)$  is in  $L_2^p[0, +\infty)$ . Similarly,  $L_2$  input-to-state stability on a set  $W$  of a state space realization implies that when  $u \in L_2^m(-\infty, 0]$  then the corresponding state vector,  $x(t)$ , (assuming initial condition  $x(-\infty) = 0$ ) is finite on  $(-\infty, 0]$  and always contained in  $W$ . In this context, consider the following definition.

**Definition 4.1** *For any causal  $L_2$ -stable input-output system  $S$ , the corresponding Hankel operator is*

$$\begin{aligned}\hat{\mathcal{H}} &: L_2^m[0, +\infty) \rightarrow L_2^p[0, +\infty) \\ &: \hat{u} \rightarrow \hat{y} = (S \circ \mathcal{F})(\hat{u}).\end{aligned}$$

Observe that the usual interpretation from linear system theory that  $\hat{\mathcal{H}}$  maps *past inputs* to *future outputs* is preserved by this definition. If  $S$  can be described by the generating series

$$y(t) = \sum_{\eta \in I^*} c(\eta) E_\eta(t, -\infty)[u],$$

then for  $t > 0$  it can be shown that

$$\hat{y}^{(i)}(t) = \sum_{\eta \in I^*} c(\underbrace{0 \dots 0}_i \eta) E_\eta(t, -\infty)[u] \quad (9)$$

when  $i \geq 0$  and  $u = \mathcal{F}(\hat{u})$ .

To see the connection between the Hankel operator and the Hankel mapping of Section 3, we proceed as earlier. In matrix notation, equation (9) becomes

$$\begin{aligned}\begin{bmatrix} \hat{y}(t) \\ \hat{y}'(t) \\ \hat{y}''(t) \\ \vdots \end{bmatrix} &= \begin{bmatrix} c(\gamma_1 \eta_1) & c(\gamma_1 \eta_2) & c(\gamma_1 \eta_3) & \dots \\ c(\gamma_2 \eta_1) & c(\gamma_2 \eta_2) & c(\gamma_2 \eta_3) & \dots \\ c(\gamma_3 \eta_1) & c(\gamma_3 \eta_2) & c(\gamma_3 \eta_3) & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \\ &\quad \underbrace{\hspace{10em}}_{\mathcal{H}_0} \\ &\quad \underbrace{\begin{bmatrix} E_{\eta_1}(t, -\infty)[u] \\ E_{\eta_2}(t, -\infty)[u] \\ E_{\eta_3}(t, -\infty)[u] \\ \vdots \end{bmatrix}}_{\mathcal{E}_\eta(t, -\infty)[u]},\end{aligned}$$

using the re-indexing  $\{\eta_i\}_{i \geq 0}$  of  $I^*$ , defining the new index set  $\{\gamma_i\}_{i \geq 0}$  with  $\gamma_{i+1} = 0\gamma_i$  and  $\gamma_1 = \emptyset$  for all  $i \geq 0$ , and defining  $\mathcal{E}_\eta(\cdot)[\cdot]$  in a manner analogous to the linear case in equation (2). Note that  $\mathcal{H}_0$  represents a special type of Hankel mapping where there is essentially no input being applied ( $u \equiv 0$ ) for positive time. That is, we can consider  $m = 0$ , and thus  $\mathcal{Z} = \{z_0\}$ , when  $t > 0$ . (This has important ramifications for zero-state observable state space realizations discussed below and in [18].) Now choose some  $t^* > 0$ , then in a neighborhood of  $t^*$  we have

$$\hat{y}(t) = \sum_{i=0}^{\infty} \frac{\hat{y}^{(i)}(t^*)}{i!} (t - t^*)^i \quad (10)$$

$$\begin{aligned}\hat{\mathcal{H}}(\hat{u}) &= \mathcal{E}_0^T(t, t^*) \hat{\mathcal{Y}}(t^*) \\ &= \mathcal{E}_0^T(t, t^*) \mathcal{H}_0 \mathcal{E}_\eta(t^*, -\infty)[u].\end{aligned} \quad (11)$$

(We shall assume hereafter that we can let  $t^* \rightarrow 0$ , and that the Taylor series (10) converges for all  $t > 0$ .)

Next let  $(f, g, h)$  be a state space realization of  $S$  with dimension  $n$  in a neighborhood  $W$  of  $x^o \in M$ . From equations (8) and (11), it follows immediately that

$$\begin{aligned}\hat{y}(t) &= \sum_{i=0}^{\infty} \sum_{\eta \in I^*} c(\underbrace{0 \dots 0}_i \eta) E_\eta(t^*, -\infty)[u] \underbrace{E_{0 \dots 0}_i(t, t^*)}_i \\ &= \sum_{i=0}^{\infty} \sum_{\eta \in I^*} L_{g_\eta} L_{g_0}^i h(x^o) E_\eta(t^*, -\infty)[u] \underbrace{E_{0 \dots 0}_i(t, t^*)}_i \\ &= \sum_{\eta \in I^*} L_{g_\eta} \left( \sum_{i=0}^{\infty} L_{g_0}^i h E_{0 \dots 0}_i(t, t^*) \right) (x^o) E_\eta(t^*, -\infty)[u].\end{aligned}$$

This suggests the factorization theorem given below whose proof relies on the following lemma adapted from [5, 7].

**Lemma 4.1** *Let  $\{g_0, g_1, \dots, g_m\}$  be a set of analytic vector fields defined on a neighborhood  $W \subset M$ . Let  $\lambda$  be an analytic vector field defined on  $W$  with an associated convergent generating series*

$$v(t) = \sum_{\eta \in I^*} L_{g_\eta} \lambda(x^o) E_\eta(t, t_0)[u],$$

for some fixed  $x^o \in W$ . Then for any  $p$ -component vector-valued analytic function  $\gamma$  defined on  $W$ , the composition  $\gamma \circ v$  has the corresponding generating series

$$w(t) = \sum_{\eta \in I^*} L_{g_\eta} (\gamma \circ \lambda)(x^o) E_\eta(t, t_0)[u].$$

**Theorem 4.1** *Let  $(f, g, h)$  be an analytic realization in a neighborhood  $W$  of 0 of an  $L_2$ -stable input-output mapping  $S : u \rightarrow y(t) = \sum_{\eta \in I^*} c(\eta) E_\eta(t, -\infty)[u]$ . If the realization is  $L_2$  input-to-state stable on  $W$  then the corresponding Hankel operator  $\hat{\mathcal{H}} : \hat{u} \rightarrow \hat{y}$  can be written as the composition*

$$\hat{\mathcal{H}} = \hat{\mathcal{O}} \circ \hat{\mathcal{C}},$$

where the controllability and observability operators are defined, respectively, as

$$\begin{aligned}\hat{C} : L_2^m[0, +\infty) &\rightarrow W_c \\ : \hat{u} \rightarrow x &= \sum_{\eta \in I^*} L_{g_\eta} \mathcal{I}(0) E_\eta(0, -\infty)[u]\end{aligned}$$

and

$$\begin{aligned}\hat{O} : W_c &\rightarrow L_2^p[0, +\infty) \\ : x \rightarrow \hat{y}(t) &= \sum_{i=0}^{\infty} L_{g_0}^i h(x) \underbrace{E_{0\dots 0}}_i(t, 0)\end{aligned}$$

with  $W_c := \hat{C}(L_2^m[0, +\infty)) \subset W$ ,  $\mathcal{I} : W \rightarrow W$  denoting the identity map on  $W$ ,  $u(t) = \mathcal{F}(\hat{u})$  and  $g_0 := f$ .

*Proof:* The  $L_2$  input-to-state stability assumption clearly guarantees that the operator  $\hat{C}$  is well defined. For any  $x \in W_c$ , there exists by definition at least one  $\hat{u} \in L_2^m[0, +\infty)$  such that  $x = \hat{C}(\hat{u})$ . Furthermore,  $(f, g, h)$  is known to be a realization of  $S$ , and  $S$  is assumed to be  $L_2$ -stable. Thus, the corresponding  $\hat{y} = \hat{\mathcal{H}}(\hat{u}) = \hat{O}(x)$  must be in  $L_2^p[0, +\infty)$ , implying that  $\hat{O}$  is well defined on  $W_c$ . Now from the definitions of  $\hat{O}$  and  $\hat{C}$  and for any  $\hat{u} \in L_2^m[0, +\infty)$ , it follows immediately that

$$(\hat{O} \circ \hat{C})(\hat{u}) = \sum_{i=0}^{\infty} L_{g_0}^i h \left( \sum_{\eta \in I^*} L_{g_\eta} \mathcal{I}(0) E_\eta(0, -\infty)[u] \right) \underbrace{E_{0\dots 0}}_i(t, 0).$$

For any fixed  $t \in [0, +\infty)$  define the mapping

$$\begin{aligned}\gamma_t : W_c &\rightarrow \mathbb{R}^p \\ : x &\rightarrow \gamma_t(x) := \sum_{i=0}^{\infty} L_{g_0}^i h(x) \underbrace{E_{0\dots 0}}_i(t, 0).\end{aligned}$$

Thus, we can write

$$(\hat{O} \circ \hat{C})(\hat{u}) = \gamma_t \left( \sum_{\eta \in I^*} L_{g_\eta} \mathcal{I}(0) E_\eta(\tau, -\infty)[u] \right) \Big|_{\tau=0}.$$

Now apply Lemma 4.1:

$$\begin{aligned}(\hat{O} \circ \hat{C})(\hat{u}) &= \sum_{\eta \in I^*} L_{g_\eta} (\gamma_t \circ \mathcal{I})(0) E_\eta(\tau, -\infty)[u] \Big|_{\tau=0} \\ &= \sum_{\eta \in I^*} L_{g_\eta} \gamma_t(0) E_\eta(0, -\infty)[u] \\ &= \sum_{\eta \in I^*} L_{g_\eta} \left( \sum_{i=0}^{\infty} L_{g_0}^i h \underbrace{E_{0\dots 0}}_i(t, 0) \right) (0) \cdot \\ &\quad E_\eta(0, -\infty)[u] \\ &= \hat{\mathcal{H}}(\hat{u}).\end{aligned}$$

This proves the theorem. ■

We finally show how to relate the generalized controllability and observability operators to the nonlinear Gramian generalizations of Scherpen. The key tool is the notion of an adjoint operator for *any* mapping between two linear spaces.

Let  $F$  be a topological vector space over  $\mathbb{R}$  with dual space  $F'$  [12]. Let  $E$  be a nonempty set and  $E^\beta$  the linear space of all real-valued functions on  $E$ . For any mapping  $T : E \rightarrow F$  define the *dual map* of  $T$  as

$$\begin{aligned}T' : F' &\rightarrow E^\beta \\ : y' &\rightarrow (T'(y'))(x) := (y' \circ T)(x), \quad x \in E\end{aligned}$$

(see, for example, [1]). Now if  $F$  is endowed with an inner product  $\langle \cdot, \cdot \rangle_F$  then it follows from the Riesz representation theorem that for any  $y' \in F'$  there exists a unique  $y \in F$  such that  $y'(\cdot) = \langle y, \cdot \rangle_F$ . Hence one can write the identity

$$(T'(y'))(x) = \langle y, T(x) \rangle_F, \quad x \in E.$$

Now suppose  $E$  has an inner product  $\langle \cdot, \cdot \rangle_E$ , and let  $y \in F$  be fixed. We are interested in the problem of determining a corresponding  $\tilde{x}_y \in E$  such that

$$\langle T(x), y \rangle_F = \langle x, \tilde{x}_y \rangle_E, \quad x \in E. \quad (12)$$

If  $T$  were a linear operator then such an  $\tilde{x}_y$  is known to always exist, in fact  $\tilde{x}_y = T^*(y)$ , where  $T^*$  is the adjoint of  $T$ . But in this more general context, the existence of  $\tilde{x}_y$  is not automatic. In fact, it is conjectured that the identity (12) may only be meaningful if  $\tilde{x}_y$  is also a function of  $x$  as well. (For more details see [18].) In what follows below, we simply assume the existence of a well defined mapping  $T^* : F \times E \rightarrow E$  such that

$$\langle T(x), y \rangle_F = \langle x, T^*(y, x) \rangle_E, \quad x \in E, \quad y \in F. \quad (13)$$

Consider a realization  $(f, g, h)$  from Theorem 4.1 with the additional assumptions that we are working in local coordinates where  $h(0) = 0$ , there is an equilibrium at 0, i.e.,  $f(0) = 0$ , and this equilibrium is asymptotically stable on  $W$ . If the realization is asymptotically reachable on  $W$  (see [16, 17]), then it is clear that for every  $x \in W$  there exists at least one  $\hat{u} \in L_2^m[0, +\infty)$  such that  $\hat{C}(\hat{u}) = x$ . The existence of a unique minimum energy control thus guarantees a well defined pseudo-inverse of  $\hat{C}$  on  $W$ , denoted here by  $\hat{C}^\dagger$ . The following identities are immediate after applying (13) with  $T = \hat{C}^\dagger$  and  $y = \hat{C}^\dagger(x)$ :

$$\begin{aligned}L_c(x) &= \frac{1}{2} \|\hat{C}^\dagger(x)\|_{L_2}^2 \\ &= \frac{1}{2} \langle \hat{C}^\dagger(x), \hat{C}^\dagger(x) \rangle_{L_2} \\ &= \frac{1}{2} \langle x, \hat{C}^{\dagger*}(\hat{C}^\dagger(x), x) \rangle \\ &:= \frac{1}{2} \langle x, p(x) \rangle.\end{aligned} \quad (14)$$

It was shown in [13, 14] that another consequence of asymptotic reachability is that  $L_c$  must always have a

local minimum at  $x = 0$ , i.e.,  $\frac{\partial L_c}{\partial x}(0) = 0$ . Thus, after differentiating the expression for  $L_c$  given in (14), it is clear that one can always write  $p(x) = \hat{P}(x)x$  for some matrix-valued function  $\hat{P}$ . The corresponding notion for  $L_o(x)$  follows analogously:

$$\begin{aligned} L_o(x) &= \frac{1}{2} \|\hat{O}(x)\|_{L_2}^2 \\ &= \frac{1}{2} \langle \hat{O}(x), \hat{O}(x) \rangle_{L_2} \\ &= \frac{1}{2} \langle x, \hat{O}^*(\hat{O}(x), x) \rangle \\ &:= \frac{1}{2} \langle x, q(x) \rangle. \end{aligned} \quad (15)$$

In this case, if the system is zero-state observable, then it is known that  $L_o$  must have a local minimum at  $x = 0$ , i.e.,  $\frac{\partial L_o}{\partial x}(0) = 0$  [13, 14]. Thus, after differentiating the expression for  $L_o$  given in (15), it follows that one can always write  $q(x) = \hat{Q}(x)x$  for some matrix-valued function  $\hat{Q}$ . Comparing the functions  $\hat{P}$  and  $\hat{Q}$  defined here to the expression given in (6) and (7), respectively, allows one to conclude that the linear case always results in the trivial situation where the functions  $\hat{P}$  and  $\hat{Q}$  are constant functions, specifically,  $\hat{P}(x) = P^{-1}$  and  $\hat{Q}(x) = Q$  for all  $x \in W$ .

#### 4. Conclusions and Future Research

In this paper, a definition of a system Hankel operator for a causal  $L_2$ -stable input-output system was introduced. Then a connection was made between this operator and the well-known Hankel mapping of Fliess through the use of a generating series representation for the input-output system. Next, a unique factorization of this Hankel operator was shown to exist given any affine state space realization with a certain stability property. Finally, direct connections to well known and new nonlinear Gramian extensions were made via the notion of a generalized adjoint operator for a nonlinear mapping.

Future research in this area includes the development of explicit representations for the energy functions in terms of a given state space realization, and new connections between the Hankel singular values functions defined in [13, 14] and the Hankel operator introduced here (see [18]). An ultimate application of this research may be the synthesis of algorithms for model reduction of nonlinear systems.

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